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Large-q asymptotics of the random-bond Potts model

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We numerically examine the large-q asymptotics of the q-state random bond Potts model. Special attention is paid to the parametrization of the critical line, which is determined by combining the loop representation of the transfer matrix with Zamolodchikov's c-theorem. Asymptotically the central charge seems to behave like $c(q)=\frac{1}{2}\log_2(q)+O(1)$. Very accurate values of the bulk magnetic exponent x_1 are then extracted by performing Monte Carlo simulations directly at the critical point. As $q\to\infty$, these seem to tend to a nontrivial limit, $x_1\to 0.192\pm 0.002$.

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I. INTRODUCTION

Recently the two-dimensional q-state random bond Potts model with $q>4$ has attracted considerable interest, because it serves as a paradigm for examining the effect of quenched randomness [1] on a first-order phase transition [2]. Since in this case the randomness couples to the local energy density, a theorem by Aizenman and Wehr [3], along with related analytical work [4,5], suggests that the transition should become continuous, as has indeed been verified by subsequent numerical studies [6–13]. Unfortunately, analytical results have been scarce, except in the limit $q\to\infty$ where properties of a particular tricritical point were related to those of the zero-temperature fixed point of the random *field* Ising model in $d=2+\varepsilon$ dimensions [8]. From the conjectured phase diagram [8] it is, however, known that this fixed point is not the analytical continuation of the line of random fixed points found for finite $q>2$ [14,15]. Namely, the latter (henceforth referred to as the $q\to\infty$ limit of the model) is rather believed to be associated with a subtle percolationlike limit [8], the exact properties of which have not yet been fully elucidated.

In the present Rapid Communication we seek to gain further knowledge of this $q\to\infty$ limit by producing numerical results along the aforementioned line of critical fixed points for very large values of q. Since cross-over effects to the pure and percolative limits of the model have been shown to be important [9,10], special attention must be paid to the parametrization of the critical line. Generalizing a recently developed transfer matrix technique [16], in which the Potts model is treated through its loop representation [17], we were able to explicitly trace out this line, and as a by-product obtain very precise values of the central charge. Based on our numerical results for the $q=8^k$ state model with $k=1,2,\ldots,6$ we find compelling evidence that

$$c(q)=\tfrac{1}{2}\log_2(q)+O(1). \tag{1.1}$$

Although this behavior of the central charge is reminiscent of the Ising-like features of the tricritical fixed point discussed above, we shall soon see that from the point of view of the magnetic exponent the $q\to\infty$ limit is most definitely not in the Ising universality class. Note also that our precision allows us to convincingly distinguish the numerically computed central charge from its analytically known value in the percolation limit [9].

With the numerically obtained parametrization of the critical disorder strength at hand we then proceed to measure the corresponding magnetic bulk scaling dimension x_1 as a function of q. The most suitable technique here is that of conventional Monte Carlo simulations. Our results lend credibility to the belief [12] that $x_1(q)$ saturates as $q\to\infty$. Based on results for the $q=8^k$ state model with $k=1,2,3$ we propose the limiting value

$$x_1(q)\to 0.192\pm 0.002 \quad \text{for} \quad q\to\infty, \tag{1.2}$$

in agreement with the one reported in Ref. [12]. The fact that Eq. (1.2) does not coincide with any known scaling dimension of standard percolation is remarkable, and calls for further analytical investigations of the $q\to\infty$ limit.

After explaining the loop model transfer matrices in Sec. II, we state our results for the critical line and the central charge in Sec. III. The Monte Carlo method and the resulting values of the magnetic scaling dimension are presented in Sec. IV, and we conclude with a discussion.

II. LOOP MODEL TRANSFER MATRICES

The partition function of the random bond Potts model can be written as

$$Z=\sum_{\{\sigma\}}\prod_{\langle ij\rangle} e^{K_{ij}\delta_{\sigma_i,\sigma_j}}, \tag{2.1}$$

where the summation is over the q discrete values of each spin and the product runs over all nearest-neighbor bonds on the square lattice. The K_{ij} are the reduced coupling constants, which for the moment may be drawn from an arbi-

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trary distribution. By the standard Kasteleyn-Fortuin transformation [18], Eq. (2.1) can be recast as a random cluster model

$$Z=\sum_{\{\mathcal{G}\}} q^{C(\mathcal{G})} \prod_{\langle ij\rangle \in \mathcal{G}} (e^{K_{ij}}-1), \tag{2.2}$$

where $\mathcal{G}$ is a bond percolation graph with $C(\mathcal{G})$ independent clusters. Note that q now enters only as a (continuous) parameter, and since the nonlocality of the clusters does not obstruct the construction of a transfer matrix [19] the interesting regime of $q \gg 4$ becomes readily accessible, provided that one can take into account the randomness in the couplings [9].

In an analogous fashion we can adapt the even more efficient loop model representation [16] to the random case. Indeed, trading the clusters for their surrounding loops on the medial lattice [17], Eq. (2.2) is turned into

$$Z=q^{N/2}\sum_{\{\mathcal{G}\}} q^{L(\mathcal{G})/2} \prod_{\langle ij\rangle \in \mathcal{G}} \left(\frac{e^{K_{ij}}-1}{\sqrt{q}}\right), \tag{2.3}$$

where N is the total number of spins, and configuration $\mathcal{G}$ encompasses $L(\mathcal{G})$ loops. The strip width L is measured in terms of the number of ''dangling'' loop segments, and must be even by definition of the medial lattice [16].

A pleasant feature of the random bond Potts model is that the critical temperature is known exactly by self-duality [20]. Employing for simplicity the bimodal distribution

$$P(K_{ij})=\tfrac{1}{2}[\,\delta(K_{ij}-K_1)+\delta(K_{ij}-K_2)\,], \tag{2.4}$$

and choosing the parametrization $s_{ij}\equiv(e^{K_{ij}}-1)/\sqrt{q}$, the self-duality criterion takes the simple form

$$s_1 s_2 = 1. \tag{2.5}$$

To fully identify the critical point the only free parameter is then the strength of the disorder, which can be measured in terms of $R\equiv K_1/K_2>1$ or $s\equiv s_1>1$.

III. CENTRAL CHARGE

In Ref. [16] we showed that Zamolodchikov's c-theorem [21] is a powerful tool for numerically identifying the fixed points of a *pure* system. The idea is simple: From the leading eigenvalue of the transfer matrix, specific free energies $f_0(L)$ can be computed as a function of the strip width L. Effective central charges $c(L)$ are then obtained by fitting data for two consecutive strip widths according to [22]

$$f_0(L)=f_0(\infty)-\frac{\pi c}{6L^2}+\cdots. \tag{3.1}$$

By tuning the free parameter s of the system, local extrema $c(L,s_*(L))$ are sought for, and finally the fixed point is identified by extrapolation: $s_*=s_*(L\to\infty)$.

In principle this strategy can also be employed for a *disordered* system, provided that error bars are carefully kept under control. Now $f_0(L)$ is related to the largest Lyapunov exponent of a product of $M\to\infty$ random transfer matrices [23,8], and its statistical error vanishes as $M^{-1/2}$ by the central limit theorem. Thus, for large enough M any desired precision on $f_0(L)$ can be achieved.

TABLE I. Effective central charge of the $q=8$ state model, as a function of disorder strength s. Two- and three-point fits to Eq. (3.1) are labeled as $C(L,L+2)$ and $C(L,L+4)$, respectively.

s	$c(4,8)$	$c(6,10)$	$c(8,12)$	$c(4,6)$	$c(6,8)$	$c(8,10)$	$c(10,12)$
3	1.495	1.500	1.500	1.4101	1.4544	1.4731	1.4821
4	1.512	1.517	1.516	1.4157	1.4657	1.4868	1.4967
5	1.519	1.525	1.523	1.4152	1.4690	1.4918	1.5025
6	1.521	1.528	1.527	1.4116	1.4683	1.4927	1.5044
7	1.520	1.529	1.529	1.4067	1.4656	1.4915	1.5041
8	1.518	1.528	1.529	1.4013	1.4619	1.4890	1.5026
9	1.509	1.530	1.528	1.3972	1.4552	1.4860	1.5004
10	1.511	1.525	1.527	1.3908	1.4534	1.4826	1.4977
11	1.501	1.526	1.526	1.3873	1.4465	1.4791	1.4949
12	1.504	1.519	1.524	1.3816	1.4451	1.4756	1.4919

An important observation is that for larger and larger L, the $c(L)$ found from Eq. (3.1) become increasingly sensitive to errors in $f_0(L)$. Therefore, M must be chosen in accordance with the largest strip width $L_{\max}$ used in the simulations. For the system at hand we found that four significant digits in $c(L)$ were needed for a reasonable precise identification of $s_*(L)$, and with $L_{\max}=12$ this in turn implies that the $f_0(L)$ must be determined with six significant digits. We were thus led to choose $M=10^8$ for $q=8$, and $M=10^9$ for larger values of q. [Incidentally, improving our results to $L_{\max}=14$ would require augmenting M by at least a factor of 100 (apart from the increased size of the transfer matrices), and since several months of computations were spent on the present project this hardly seems possible in a foreseeable future.]

Data collection was done by dividing the strip into M/l patches of length $l=10^5$ lattice spacings, and for each patch the couplings were randomly generated from a *canonical* ensemble, i.e., the distribution (2.4) was restricted to produce an equal number of strong and weak bonds.

In the right part of Table I we show the resulting two-point fits (3.1) in the $q=8$ state model, as a function of s. The left part of the table provides analogous three-point fits, obtained by including a nonuniversal $1/L^4$ correction in Eq. (3.1). In all cases the error bars are believed to affect only the last digit reported. The two-point fits give clear evidence of a maximum in the central charge, and we estimate its location as $s_*=6.5\pm2.0$. The corresponding central charge is estimated from the three-point fits, as these are known to converge faster in the $L\to\infty$ limit [9], and we arrive at $c=1.530\pm0.001$. To appreciate the precision of this result, we mention that the numerical values of $c(q=8)$ first reported were 1.50 ± 0.05 [7] and 1.517 ± 0.025 [8].

Table II summarizes our results for other values of q. Two remarkable features are apparent. First, $s_*\propto q^{w}$ is well fitted by a power law with $w=0.31\pm0.02$. This gives valuable information on how the $q\to\infty$ limit of the model is approached, and implies that the ratio of the coupling constants $R\equiv K_1/K_2=\log(1+s\sqrt{q})/\log(1+\sqrt{q}/s)$ is a nonmonotonic function of q that tends to the *finite* limiting value $(1+2w)/(1-2w)=4.3\pm0.6$ as $q\to\infty$. We shall discuss this

TABLE II. Critical disorder strength s and central charge c, as functions of q.

q	s_*	c	$c/\log_2(q)$
8	6.5 (20)	1.530 (1)	0.5100 (3)
64	15.5 (20)	3.050 (3)	0.5083 (5)
512	32 (2)	4.545 (10)	0.5050 (11)
4096	65 (8)	6.038 (24)	0.5032 (20)
32768	135 (20)	7.54 (3)	0.5027 (20)
262144	250 (50)	9.04 (3)	0.502 (2)

finding further in Sec. V. Second, the central charge seems to fulfill the relation (1.1) as stated in the Introduction.

IV. MAGNETIC SCALING DIMENSION

In this section we explain the Monte Carlo method used for obtaining values of the magnetic scaling exponent. Simulations were performed on square lattices of size $L\times L$ with periodic boundary conditions, with L ranging from 4 to $L_{\max}=128$ for $q=8$, 64 and $L_{\max}=64$ for $q=512$.

We employed the Wolff cluster algorithm [24]. The first part of the simulations was to determine the autocorrelation times τ, which were found to increase with the lattice size and also with q. For the largest simulated lattices, we determined τ as follows: 88 ± 4 cluster updates for $q=8$ and $L=128$, 3000 ± 215 for $q=64$ and $L=128$, and $31\,000\pm 3000$ for $q=512$ and $L=64$. This rapid increase of τ with q explains why we simulate only up to $L=64$ for the largest q.

Next, we measure the magnetization, defined for each disorder sample x by

$$m_x=\frac{q\langle\rho\rangle-1}{q-1}, \tag{4.1}$$

where $\rho=\max(N_1,N_2,\ldots,N_q)/L^2$ and N_σ is the number of Potts spins taking the value σ. Here $\langle\ldots\rangle$ denotes the thermal average. Then the magnetization $m(L)$ is obtained by averaging over 10^5 disorder configurations for $q=8$, and 10^4 configurations for $q=64$ and 512. For each disorder sample, 100τ updates were dedicated to the thermalization, and a further 100τ to the magnetization measurement. Error bars were computed from the disorder fluctuations (it can easily be checked [10] that the contribution from *thermal* fluctuations is negligible), and the strength of the disorder was chosen as indicated in Table II.

From a fit to $m(L)\simeq L^{-x_1}$, we obtain for the magnetic scaling dimension

$$x_1=\begin{cases} 0.1535(10) & \text{for} \quad q=8 \\ 0.172(2) & \text{for} \quad q=64 \\ 0.180(3) & \text{for} \quad q=512\,. \end{cases} \tag{4.2}$$

We see that the magnetic exponent seems to saturate as we increase q. In view of the result (1.1) for the central charge we expect the asymptotic behavior should involve $\log(q)$ rather than q itself, and indeed the data are well fitted by

$$x_1(q)=a+b/\log(q), \tag{4.3}$$

with $a=0.192(2)$ and $b=-0.080(4)$. Thus, based on the form (4.3) we are led to propose the limiting value (1.2) of x_1 given in the Introduction.

V. DISCUSSION

It is useful to juxtapose our findings on the large-q behavior of the critical line with the phase diagram proposed in Ref. [8]. In that work the disorder strength was parametrized through $s=q^w$ with $w>0$, and the limit $w\to\infty$ was identified with classical percolation on top of the strong bonds. Actually it is easily seen from Eq. (2.2) that directly at $q=\infty$ this percolation scenario holds true whenever $w>\frac{1}{2}$, and assuming that the line of critical fixed points is described by a monotonic function $w_*(q)$ it can thus be confined to the region $w\leqslant\frac{1}{2}$. With this slight reinterpretation, Ref. [8] argues that at $q=\infty$ the critical point is located in the limit $w\to\frac{1}{2}$. Indeed, since for $q=\infty$ any initial $w\ll\frac{1}{2}$ will be driven to larger values due to mapping to the random field Ising model, this is nothing but the usual assumption of ''no intervening fixed points.''

However, this seems at odds with the results of Table II, where we found that for $q\geqslant 4$ the critical line, when measured in terms of w, saturates at $w=0.31\pm 0.02$. Unless our numerical method is flawed by some gross systematic error, it is thus *a priori* difficult to see how this can be reconciled with the above result of $w_*(q=\infty)=\frac{1}{2}$. A possible explanation is that the limits $q\to\infty$ and $w\to\frac{1}{2}$ are highly noncommuting. This is witnessed by the jump in the central charge, which in the percolation limit ($w=\infty$ and $q<\infty$) reads [8]

$$c_{\rm perc}=\frac{5\sqrt{3}}{4\pi}\ln(q)\simeq 0.477\,69\log_2(q), \tag{5.1}$$

to be contrasted with our numerical result (1.1).

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